JOURNAL OF APPROXIMATION THEORY 57, 259–267 (1989)

Small Chebyshev Systems Made by Products*

IDO EFRAT

School of Mathematical Sciences, Tel-Aviv University, Tel-Aviv, Israel

Communicated by E. W. Cheney

Received January 28, 1986; revised January 8, 1987

We characterize the sets $\mathbf{F} = \{f_0, ..., f_n\}$ of real continuous functions for which $\mathbf{F}^2 = \{f_i f_j : 0 \leqslant i, j \leqslant n\}$ has less than $3n$ elements and the Chebyshev systems of the form $\mathbf{F}^2$ of degree less than $3n$. This extends results of Granovsky and Passow and a number-theoretic result of Freiman.

I. INTRODUCTION

In the theory of experimental designs the matrix $M(\xi) = \|m_{ij}\|_{i,j=0}^n$ plays an important role, where $m_{ij} = \int_X f_i(x) f_j(x) \xi(dx)$, $f_0, ..., f_n$ (the regression functions) are $n+1$ continuous functions on the compact space X, and ξ (the design) is a probability measure on X. Statistical considerations direct one's interest to those ξ for which $\det M(\xi)$ is maximal. Such measures are called (D-)optimal. It can be easily seen that if the spectrum of ξ concentrates at less than $n+1$ points, then $\det M(\xi) = 0$ [4, pp. 323–324]. Kiefer and Wolfowitz [5] considered the sets of continuous functions $\mathbf{F} = \{f_0, ..., f_n\}$ for which there exists an optimal design ξ_0 whose spectrum concentrates at nearly $n+1$ points. The supporting hyperplane argument of [4, pp. 330–333] yields that if $X = [\alpha, \beta]$, if $1 \notin \mathbf{F}^2 = \{f_i f_j : 0 \leqslant i, j \leqslant n\}$, and if $\{1\} \cup \mathbf{F}^2$ is a Chebyshev system of minimal degree $2n+2$ then there exists such ξ_0 with exactly $n+1$ points in its spectrum. More generally, if $1 \notin \mathbf{F}^2$ and if $\{1\} \cup \mathbf{F}^2$ is a Chebyshev system of degree $2n+s$ then there exists an optimal design which concentrates at not more than $n + \lfloor (s+1)/2 \rfloor$ points. Granovsky and Passow [3] have characterized all sets $\mathbf{F}$ for which $|\mathbf{F}^2|$ is minimal and all Chebyshev systems of the form $\mathbf{F}^2$ with minimal degree $2n+1$. A related result was obtained by Granovsky in [2]. Here we extend the results of [3] to all sets $\mathbf{F}$ for which $|\mathbf{F}^2| < 3n$ and to Chebyshev systems of the form $\mathbf{F}^2$ with degree less than $3n$. This will be done by applying a number-theoretic result of Freiman. As a consequence

* The contents of this paper constitute the author's M.Sc. thesis written under the direction of Professor G. A. Freiman.

0021-9045/89 $3.00

it will be possible to describe the Chebyshev systems of the form $\{1\} \cup \mathbf{F}^2$ with degree at most $3n$, when $1 \notin \mathbf{F}^2$.

II. The Main Results

For a subset $\mathbf{K}$ of an abelian group we define $2\mathbf{K} = \{a+b: a, b \in \mathbf{K}\}$. Freiman has proved [1, pp. 11–14] that if $\mathbf{K} = \{a_0, ..., a_n\}$ is a set of integers and if $|2\mathbf{K}| = 2n+b$ with $1 \leqslant b < n$, then $\mathbf{K}$ is contained in an arithmetical progression of length $n+b$. Note that always $|2\mathbf{K}| \geqslant 2n+1$. We first generalize this to sets of real numbers:

Proposition. *Let $\mathbf{K} = \{a_0, ..., a_n\}$ be a set of real numbers and suppose that $a_0 = 0$, $1 = a_1 < \cdots < a_n$. If some a_i is irrational then $|2\mathbf{K}| \geqslant 3n$.*

Proof. By induction on n. For $n=2$, $\mathbf{K} = \{0, 1, a_2\}$ with a_2 irrational and we have $2\mathbf{K} = \{0, 1, 2, a_2, 1+a_2, 2a_2\}$. Obviously, these are six distinct numbers. Suppose now that $n \geqslant 3$ and that the assertion is true for sets with n elements. Let $\mathbf{K}$ be as above and let a_i be the first irrational in $\mathbf{K}$.

Case i. For some $1 \leqslant j \leqslant n$, $(a_j-1)/(a_2-1)$ is irrational. In this case, let $\mathbf{K}' = \{a_1, ..., a_n\}$ and let $\mathbf{K}'' = (\mathbf{K}'-1)/(a_2-1)$. By the induction hypothesis $|2\mathbf{K}'| = |2\mathbf{K}''| \geqslant 3n-3$. Also, $2\mathbf{K} \setminus 2\mathbf{K}'$ contains $0, 1$, and a_i. Therefore $|2\mathbf{K}| \geqslant 3n$.

Case ii. $(a_j-1)/(a_2-1)$ is rational for $j = 1, ..., n$. Then for all such j,

$$\frac{a_n - a_j}{a_n - a_{n-1}} = \left[\frac{a_n-1}{a_2-1} - \frac{a_j-1}{a_2-1}\right] \cdot \left[\frac{a_n-1}{a_2-1} - \frac{a_{n-1}-1}{a_2-1}\right]^{-1}$$

is rational. In particular $(a_n-1)/(a_n-a_{n-1})$ is rational. Now assume that $a_n/(a_n-a_{n-1})$ is rational too. Then so is a_n-a_{n-1} and thus, is so a_n-a_j for $j=1, ..., n$. By taking $j=1$ and then $j=i$ we get a contradiction. Hence $a_n/(a_n-a_{n-1})$ is irrational. Set $\mathbf{K}''' = (a_n-\mathbf{K})/(a_n-a_{n-1})$. Since $n \geqslant 3$, $\mathbf{K}'''$ satisfies the requirements of Case i with $j=n$, so we obtain: $|2\mathbf{K}| = |2\mathbf{K}'''| \geqslant 3n$. This completes the proof.

The inequality $|2\mathbf{K}| \geqslant 3n$ in the proposition cannot be improved, as can be seen by examining the set $\mathbf{K} = \{0, 1, 2, ..., n-1, \sqrt{2}\}$ for which $|2\mathbf{K}| = 3n$.

From the proposition and Freiman's cited result we obtain:

Corollary. *Let $\mathbf{K} = \{a_0, ..., a_n\}$ be a set of real numbers such that $|2\mathbf{K}| = 2n+b$, where $1 \leqslant b < n$. Then $\mathbf{K}$ is contained in an arithmetical progression of length $n+b$.*

This result will be generalized further in the following theorem, where we consider the multiplicative structure of sets $\mathbf{F}=\{f_0, ..., f_n\}$ of real-valued functions defined and continuous on a closed interval $[\alpha, \beta]$. There, in addition to the requirement that $|\mathbf{F}^2|=2n+b$, where $1 \leqslant b < n$, one has to assume, as in [3], that the set

$$A=\{x \in [\alpha, \beta]: f_0(x), ..., f_n(x) \text{ are nonzero, have distinct absolute values, and } |\{|f_0(x)|, ..., |f_n(x)|\}^2|=2n+b\} \tag{1}$$

is large enough. Then $\mathbf{F}$ is contained in a short geometric progression:

THEOREM 1. *Let* $\mathbf{F}=\{f_0, ..., f_n\}$, $1 \leqslant b < n$, *and* A *be as above. If* $|\mathbf{F}^2|=2n+b$ *and if* A *has a discrete complement in* $[\alpha, \beta]$, *then there exists a set* $\mathbf{S}=\{s_0, ..., s_n\}$ *and real-valued functions* w *and* u *such that:*

(i) $f_i(x)=w(x)u(x)^{s_i}$, $i=0, 1, ..., n$, *whenever the term on the right is defined.*

(ii) $\mathbf{S} \subseteq \{0, 1, ..., n+b-1\}$, $|2\mathbf{S}|=2n+b$, $\min \mathbf{S}=0$.

(iii) w *is defined and continuous on* $[\alpha, \beta]$.

(iv) u *is defined and continuous whenever* $w(x) \neq 0$.

(v) *For* $x \in A$, $w(x) \neq 0$ *and* $u(x) \neq 0, 1$.

(vi) *If* $w(x_0)=0$ *then* $\lim_{x \to x_0} w(x)u(x)^{\max \mathbf{S}}$ *exists and is finite.*

Note that the converse of Theorem 1 also holds: if A' is a subset of $[\alpha, \beta]$ with discrete complement, if $1 \leqslant b < n$ and if $\mathbf{S}=\{s_0, ..., s_n\}$, w and u satisfy (ii)–(vi) (with A replaced by A'), then, for each $0 \leqslant i \leqslant n$, wu^{s_i} can be (uniquely) extended to a continuous function f_i on $[\alpha, \beta]$ such that $|\mathbf{F}^2|=2n+b$, with $\mathbf{F}=\{f_0, ..., f_n\}$.

We will use the following lemmas:

LEMMA 1. *Let* $r \geqslant 2$, *let* $\mathbf{S}=\{s_0, ..., s_n\}$ *be a set of integers, at least two of which are consecutive, and suppose that* $0=s_0< \cdots <s_n$. *If whenever* $s_j-s_i=r$, $j=i+1$, *and if such a pair* i, j *exists, then* $|2\mathbf{S}| \geqslant 3n$.

Proof. For $n=2$ the assertion is clear. Suppose it holds for sets with a smaller number of elements but fails for $\mathbf{S}$. Denote $\mathbf{S}'=\{s_0, ..., s_{n-1}\}$. By considering, if necessary, $s_n-\mathbf{S}$ instead of $\mathbf{S}$, we may assume that $\mathbf{S}'$ also contains a gap of length r. Also, $\mathbf{S}'$ must contain at least two consecutive integers, for otherwise $s_n=s_{n-1}+1$, and $0, s_1, ..., s_n, s_n+s_1, ..., 2s_n, s_{n-1}+s_1, ..., 2s_{n-1}$ are $3n$ distinct elements of $2\mathbf{S}$. By the induction hypothesis, $|2\mathbf{S}'| \geqslant 3n-3$. Also, $2\mathbf{S} \setminus 2\mathbf{S}'$ includes $s_{n-1}+s_n$ and $2s_n$. Since $|2\mathbf{S}|<3n$, $2\mathbf{S}=2\mathbf{S}' \cup \{s_{n-1}+s_n, 2s_n\}$. We will show now that for all $0 \leqslant i \leqslant n$,

$$s_i \equiv s_n \pmod{s_n-s_{n-1}}. \tag{2}$$

For $i=n, n-1$ this is clear. Suppose that $i \leqslant n-2$ and that (2) is valid for $i+1, ..., n$. Consider s_i+s_n. It is an element of 2S which is smaller than $s_{n-1}+s_n$ and $2s_n$. Hence, it belongs to 2S′, that is, there exist $k, l \leqslant n-1$ with $s_i+s_n=s_k+s_l$. It can be easily seen that $i<k, l$ and obviously, $s_i-s_n=s_k+s_l-2s_n$. By our assumption s_k, s_l, s_n are all congruent mod s_n-s_{n-1}, and therefore (2) holds also for i. Since S contains two consecutive integers we must have $s_n=s_{n-1}+1$. Now let $i_1, ..., i_m$ be a list of all $0 \leqslant i<n$ for which $s_{i+1}=s_i+r$. Then 2S contains the following elements:

$$s_0, ..., s_{i_1-1}$$

$$s_{i_1}+\mathbf{S}$$

$$s_{i_1+1}+\mathbf{S}$$

$$s_{i_1+2}+s_n, \qquad s_{i_1+3}+s_n, ..., 2s_n$$

$$s_{i_2+1}+s_{n-1}, \qquad s_{i_3+1}+s_{n-1}, ..., s_{i_m+1}+s_{n-1}.$$

The only elements of 2S which appear in this list more than once are $s_{i_1}+s_{i_1+1}, ..., s_{i_1}+s_{i_m+1}$ which appear twice. Hence,

$$|2\mathbf{S}| \geqslant i_1+2(n+1)-m+(n-i_1-1)+(m-1)=3n$$

contrary to the assumption on S.

LEMMA 2. *Let* $\mathbf{K}=\{a_0, ..., a_n\}$ *be a subset of* $\mathbb{Z} \oplus G$ *where* G *is an abelian group, with* $a_i=(m_i, \alpha_i)$ *and* $m_0<\cdots<m_n$. *If* $|2\mathbf{K}|<3n$ *then* $\alpha_0, ..., \alpha_n$ *belong to a translate of some cyclic subgroup of* G.

Proof. We use induction on n. For $n=1$ there is nothing to prove. For $n \geqslant 2$ suppose that $|2\mathbf{K}|<3n$ but that $\alpha_0, ..., \alpha_n$ do not belong to any translate of a cyclic subgroup of G. Set $\mathbf{K}'=\{a_0, ..., a_{n-1}\}$.

Case i. $\alpha_0, ..., \alpha_{n-1}$ belong to a translate H of a cyclic subgroup of G. Then $\alpha_n \notin H$. 2K contains 2K′, $\mathbf{K}'+\{a_n\}$, and $2a_n$. Since $m_i<m_n$ for all $0 \leqslant i<n$, $2a_n \notin 2\mathbf{K}' \cup (\mathbf{K}'+\{a_n\})$. Also, if for some $0 \leqslant i, j, k<n$, $a_i+a_j=a_k+a_n$ then $\alpha_n=\alpha_i+\alpha_j-\alpha_k \in H$ which is a contradiction. Thus, 2K′ and $\mathbf{K}'+\{a_n\}$ are disjoint. Therefore, $|2\mathbf{K}| \geqslant |2\mathbf{K}'|+|\mathbf{K}' \cup \{a_n\}|+1 \geqslant 2n-1+n+1=3n$, contrary to the assumption.

Case ii. $\alpha_0, ..., \alpha_{n-1}$ do not all belong to any translate of a cyclic subgroup of G. By the induction hypothesis, $|\mathbf{K}'| \geqslant 3n-3$. However, $2\mathbf{K} \backslash 2\mathbf{K}'$ contains $2a_n$ and a_n+a_{n-1}. Since $|2\mathbf{K}|<3n$ we obtain $2\mathbf{K}=2\mathbf{K}' \cup \{2a_n, a_n+a_{n-1}\}$. Consequently, for each $0 \leqslant i \leqslant n-2$ there exist $0 \leqslant j, k \leqslant n-1$ such that $a_i+a_n=a_j+a_k$, so $\alpha_i=\alpha_j+\alpha_k-\alpha_n$. An inductive argument yields that $\alpha_i \in \alpha_n+\langle\alpha_{n-1}-\alpha_n\rangle$ (this clearly holds for $i=n-1$ and $i=n$), and we get a contradiction.

LEMMA 3. *If* $\mathbf{K}=\{a_0, ..., a_n\}$, $0<|a_0|<\cdots<|a_n|$, *if* $|\mathbf{K}^2|<3n$, *and if* $\log|a_i|=q+ps_i$ *where* $p>0$ *and* $\mathbf{S}=\{s_0, ..., s_n\}$ *is a set of integers, at least two of which are consecutive, then for all* $0\leqslant i, j\leqslant n$:

$$s_i\equiv s_j(\text{mod } 2)\Rightarrow \text{sg } a_i=\text{sg } a_j.$$

Proof. The mapping $\theta(a_i)=(s_i, (1-sga_i)/2, (1-(-1)^{s_i})/2)$ is an isomorphism of $\mathbf{K}$ onto a subset of $\mathbb{Z}\oplus\mathbb{Z}_2\oplus\mathbb{Z}_2$ in the sense of [1, pp. 2–4], where $\mathbf{K}$ is considered to be a subset of the multiplicative group $\mathbb{R}\setminus\{0\}$. If $|2\theta(\mathbf{K})|=|\mathbf{K}^2|<3n$ then Lemma 2 yields that $\{(\text{sg } a_i, (-1)^{s_i}): i=0, ..., n\}$ has at most two elements. Since $\mathbf{S}$ contains at least two consecutive integers, the conclusion of the lemma follows.

Proof of Theorem 1. For each $x\in A$ and for each $0\leqslant i\leqslant n$ set $g_i(x)=\log|f_i(x)|$. By the corollary we can find $p(x)>0$ and $q(x)$ together with a set $\mathbf{S}_x=\{s_0(x), ..., s_n(x)\}$ of $n+1$ integers such that $\min\mathbf{S}_x=0$, $\max\mathbf{S}_x\leqslant n+b-1$, and $g_i(x)=q(x)+p(x)\,s_i(x)$ for $i=0, 1, ..., n$. Since $b<n$, $\mathbf{S}_x$ must contain a pair of consecutive integers. Since $|\mathbf{F}^2|=|\{|f_0(x)|, ..., |f_n(x)|\}^2|=2n+b$, we also have $|\{f_0(x), ..., f_n(x)\}^2|=2n+b$. Therefore by Lemma 3, $\text{sg} f_i(x)=\text{sg} f_j(x)$ whenever $s_i(x)\equiv s_j(x)$ (mod 2). Hence there exist $\varepsilon_1(x)\in\{1, -1\}$ and $\varepsilon_2(x)\in\{0, 1\}$ such that for every $0\leqslant i\leqslant n$, $\text{sg} f_i(x)=\varepsilon_1(x)(-1)^{\varepsilon_2(x)s_i(x)}$.

Now, for every $x, x'\in A$ and every i, j, k, l we must have

$$|f_i(x)f_j(x)|=|f_k(x)f_l(x)|\Leftrightarrow|f_i(x')f_j(x')|=|f_k(x')f_l(x')|$$

and thus: $s_i(x)+s_j(x)=s_k(x)+s_l(x)\Leftrightarrow s_i(x')+s_j(x')=s_k(x')+s_l(x')$. Hence

$$s_i(x)-s_k(x)=s_l(x)-s_j(x)\Leftrightarrow s_i(x')-s_k(x')=s_l(x')-s_j(x'). \tag{3}$$

Now, as was previously observed, the set $\mathbf{S}_x$ (and similarly $\mathbf{S}_{x'}$) contains at least one pair of consecutive integers. Let a be the difference $s_i(x')-s_j(x')$ where i, j satisfy $s_i(x)-s_j(x)=1$. According to (3), a is well defined. We will prove now by induction on $r\geqslant 1$ that

$$s_i(x)-s_j(x)=r\Rightarrow s_i(x')-s_j(x')=ra. \tag{4}$$

The case $r=1$ is clear. Suppose that $r\geqslant 2$ and that (4) is valid for $1, ..., r-1$. If $s_i(x)-s_j(x)=r$ then by Lemma 1 we may assume that there exists k for which $s_j(x)<s_k(x)<s_i(x)$. By the induction hypothesis

$$\begin{aligned}s_i(x')-s_j(x')&=(s_i(x')-s_k(x'))+(s_k(x')-s_j(x'))\\&=(s_i(x)-s_k(x))\,a+(s_k(x)-s_j(x))a=(s_i(x)-s_j(x))\,a\end{aligned}$$

and (4) is thus proved. Knowing (4) and knowing that $\mathbf{S}_{x'}$ contains a pair of consecutive integers we conclude that $a=1$. Also $\min \mathbf{S}_x = \min \mathbf{S}_{x'} = 0$ so $s_i(x) = s_i(x')$ for $i=0, ..., n$. Since x and x' were arbitrary distinct numbers in A, $s_i = s_i(x)$ is independent of the choice of x. Therefore $g_i(x) = q(x) + s_i p(x)$ on A. It follows that for $x \in A$,

$$f_i(x) = |f_i(x)| \cdot \operatorname{sg} f_i(x) = e^{q(x) + s_i p(x)} \cdot \varepsilon_1(x) \cdot (-1)^{s_i \varepsilon_2(x)}. \tag{5}$$

Let i, j, k be such that $s_i = 0$, $s_j - s_k = 1$. Define $w(x) = f_i(x)$, $u(x) = f_j(x)/f_k(x)$. For $x \in A$ (5) implies that $w(x)u(x)^{s_l} = f_l(x)$ for $l = 0, ..., n$. If $f_i(\bar{x}) \neq 0$ while $f_k(\bar{x}) = 0$ then by (5), $e^{q(x)} > \delta > 0$ in a neighbourhood of $\bar{x}$, and $e^{s_k p(x)} \to_{x \to \bar{x}, x \in A} 0$. Consequently, $p(x) \to_{x \to \bar{x}, x \in A} -\infty$ and therefore $u(x) = f_j(x)/f_k(x) = e^{p(x)}(-1)^{\varepsilon_2(x)} \to_{x \to \bar{x}} 0$. Hence we may extend u continuously to $\{x \in [\alpha, \beta] : w(x) \neq 0\}$ and still have $w(x)u(x)^{s_l} = f_l(x)$. (i)–(vi) can now be easily verified.

Remarks. (1) The inequality $|2\mathbf{S}| \geqslant 3n$ in Lemma 1 cannot be improved. To see this take $\mathbf{K} = \{0, 1, ..., r-2, \quad r-1, \quad 2r-1, 2r, ..., 3r-3, 3r-2\}$. Also, the value $3n$ in Lemma 3 is the best possible as can be seen by examining $\mathbf{K} = \{1, 2, 4, ..., 2^{n-1}, -2^n\}$.

(2) There exist sets $\mathbf{F} = \{f_0, ..., f_n\}$ as in Theorem 1 such that for each representation $f_i(x) = w(x)u(x)^{s_i}$ as in (1), u is discontinuous. For example, consider $[-2, 2]$, $b=1$, and $f_i(x) = x^{n-i}(1+x)^i$ for $i = 0, ..., n$. Since $f_i(1/2) = 3^i/2^n$ either $u = f_0/f_1 = x/(1+x)$ or $u = f_1/f_0 = 1 + 1/x$.

(3) The case $b=1$ of Theorem 1 was proved by Granovsky and Passow [3]. The minimal case $b=1$ of the following theorem was also proved by them. Note, however, that an inaccuracy occurs in their proof in regard to the possibility that u is discontinuous. The example considered in the previous remark shows that this can actually happen even when $\mathbf{F}^2$ is a Chebyshev system of degree $2n+1$.

Definition [6]. A set $\mathbf{T} = \{t_0, ..., t_m\}$ of natural numbers, with $t_0 < \cdots < t_n$, has the alternating parity property (APP) if for each $1 \leqslant i \leqslant n-1$, $t_{i+1} - t_i$ is odd.

Theorem 2. *Let* $\mathbf{F} = \{f_0, ..., f_n\}$ *be a set of real functions defined and continuous on* $[\alpha, \beta]$ *and let* $1 \leqslant b < n$. *If* $\mathbf{F}^2$ *is Chebyshev system of degree* $2n+b$ *then there exist a set* $\mathbf{S} = \{s_0, ..., s_n\}$ *and real valued functions w and u such that*:

(i) $f_i(x) = w(x)u(x)^{s_i}$, $i = 0, ..., n$, *whenever the term on the right is defined.*

(ii) $\mathbf{S} \subseteq \{0, 1, ..., n+b-1\}$, $|2\mathbf{S}| = 2n+b$, $\min \mathbf{S} = 0$, $|\mathbf{S}| = n+1$.

(iii) *w is continuous in* $[\alpha, \beta]$ *and vanishes at most once.*

(iv) *u is defined and continuous whenever $w \neq 0$ and is injective.*

(v) *$w(x) \neq 0$ and $|u(x)| \neq 0, 1$ on A.*

(vi) *If $w(\bar{x}) = 0$ then* $\lim_{x \to \bar{x}} |u(x)| = \infty$ and $\lim_{x \to \bar{x}} w(x) u(x)^{\max \mathbf{S}}$ *exists, is finite, and is nonzero.*

(vii) *If $w(\bar{x}) = 0$ and $\alpha < \bar{x} < \beta$ then* $\lim_{x \to \bar{x}^-} u(x) = -\lim_{x \to \bar{x}^+} u(x)$ $(= \pm\infty)$.

(viii) *If* $2\mathbf{S}$ *does not have the* APP, *then u is one-signed and $w(x) \neq 0$ in (α, β).*

Conversely, if $\mathbf{S} = \{s_0, ..., s_n\}$, *and w and u satisfy* (ii)–(viii), *then for each $0 \leqslant i \leqslant n$, wu^{s_i} can be (uniquely) extended to a continuous function f_i such that $\mathbf{F}^2 = \{f_0, ..., f_n\}^2$ is a Chebyshev system of degree $2n + b$ on $[\alpha, \beta]$.*

Proof. Clearly if $\mathbf{F}^2$ is a Chebyshev system of degree $2n+b$ then $[\alpha, \beta] \setminus A$ is finite. Let $\mathbf{S}$, w, u be as in Theorem 1. At each point $x_0 \in [\alpha, \beta]$ at least one f_i does not vanish. For if $f_0(x_0) = \cdots = f_n(x_0) = 0$ we could choose $2n + b - 1$ distinct points $x_1, ..., x_{2n+b-1}$ in A which are different from x_0, and then the following system of $2n + b - 1$ linear equations in the $2n + b$ unknowns $\{a_g : g \in \mathbf{F}^2\}$ would have a nontrivial solution

$$\sum_{g \in \mathbf{F}^2} a_g \, g(x_i) = 0 \qquad (i = 1, ..., 2n + b - 1).$$

This would give a nontrivial combination of the functions of $\mathbf{F}^2$ with $2n + b$ solutions $x_0, x_1, ..., x_{2n+b-1}$ in contradiction to $\mathbf{F}^2$ being a Chebyshev system of degree $2n + b$.

Now suppose there were $x_1, x_2 \in [\alpha, \beta]$, $x_1 \neq x_2$, with $w(x_1), w(x_2) \neq 0$ and $u(x_1) = u(x_2)$. Then we could choose distinct $x_3, ..., x_{2n+b}$ (other than x_1, x_2) in A and get a nontrivial solution for the linear system

$$\sum_{t \in 2\mathbf{S}} b_t w^2(x_i) \, u(x_i)^t = 0 \qquad (i = 2, ..., 2n + b).$$

But then, this would also hold for $i = 1$, in contradiction to the assumptions. Therefore, u is injective in $\{x \in [\alpha, \beta] : w(x) \neq 0\}$. Suppose $w(\bar{x}) = 0$. Since the functions $f_0, ..., f_n$ do not all vanish at $\bar{x}$ and since $f_i(\bar{x}) = \lim_{x \to \bar{x}} w(x) u(x)^{s_i}$ we must have

$$\lim_{x \to \bar{x}} |u(x)| = \infty.$$

Since u is injective this implies that the one-sided limits of $u(x)$ as x approaches $\bar{x}$ are ∞ and $-\infty$ (unless, of course, $\bar{x} = \alpha$ or $\bar{x} = \beta$). Again, since u is injective and continuous, there is at most one such point. Now $\mathbf{F}^2$ is a Chebyshev system of degree $2n + b$ if and only if det $\|w^2(x_i)u(x_i)^{t_j}\|_{0\leq i,j\leq m}\neq 0$ for all distinct $x_0, ..., x_m$, where $\mathbf{T}=\{t_0, ..., t_m\}=2\mathbf{S}$, $m=2n+b-1$. Note that $w^2 \cdot u^{t_j}$ is meaningful even at $\bar{x}$. If u is continuous then always $w\neq 0$, so this is equivalent to det $\|u(x_i)^{t_j}\|_{0\leq i,j\leq m}\neq 0$ for all distinct $x_0, ..., x_m$. On the other hand, if u has discontinuity at $\bar{x}$ then the above condition is equivalent to the non-vanishing of det $\|u(x_i)^{t_j}\|_{0\leq i,j\leq m}$ and of det $\|u(x_i)^{t_j}\|_{0\leq i,j\leq m-1}$ for distinct $x_0, ..., x_m$ $(\neq \bar{x})$. But this depends only on the range of u. Moreover, since the determinant is a homogeneous function of its columns, we only need to know whether u is bounded, whether it vanishes, and whether it changes sign. Therefore our problem can be reduced to the vanishing properties of $D_{\mathbf{T}}$ and $D_{\mathbf{T}\setminus\{t_m\}}$ where in general, for $\mathbf{R}=\{0=r_0<\cdots<r_m\}$, $D_{\mathbf{R}}=\det\|x_i^{r_j}\|_{0\leq i,j\leq m}$, and this is equivalent to the problem of deciding whether $\{x^r: r\in\mathbf{R}\}$ is a Chebyshev system on $\mathbb{R}$ or $\mathbb{R}\setminus\{0\}$. Passow has proved [6] that $\{x^r: r\in\mathbf{R}\}$ is a Chebyshev system on $\mathbb{R}$ if and only if $\mathbf{R}$ has the APP. His proof can also be used to show that $\mathbf{R}$ has the APP if and only if $\{x^r: r\in\mathbf{R}\}$ is a Chebyshev system on $\mathbb{R}\setminus\{0\}$ too. Now, if $\mathbf{T}$ does not have the APP then by the above discussion $\{x^t: t\in\mathbf{T}\}$ is not a Chebyshev system on $\mathbb{R}\setminus\{0\}$ and therefore $D_{\mathbf{T}}$ vanishes for some distinct and nonzero $x_0, ..., x_m$. We obtain that u must be one-signed in $[\alpha\,\beta]$. The other requirements now follow easily.

The opposite direction follows from the remark after the statement of Theorem 1, from [6], and from the well-known fact that for distinct positive $x_0, ..., x_m$ and for $0=t_0<t_1<\cdots<t_m$, det $\|x_i^{t_j}\|_{0\leq i,j\leq m}\neq 0$ [4, pp. 9–10].

Remark. When b is even, since min $2\mathbf{S}=0$ and max $2\mathbf{S}$ are even, $2\mathbf{S}$ does not have the APP.

III. Chebyshev Systems of the Form $\{1\}\cup\mathbf{F}^2$

As was mentioned in the introduction, Chebyshev systems of the form $\{1\}\cup\mathbf{F}^2$ are also of particular interest. So suppose $\mathbf{F}=\{f_0, ..., f_n\}$, $1\notin\mathbf{F}^2$, and suppose that $\{1\}\cup\mathbf{F}^2$ is a Chebyshev system on $[\alpha, \beta]$ with degree at most $3n$ so that $[\alpha, \beta]\setminus A$ is finite, where A is as before. Since $|\mathbf{F}^2|=2n+b$ with $1\leq b<n$, we obtain $\mathbf{S}$, w, u as in Theorem 1(a). An argument similar to the one used in the proof of Theorem 2 yields that $f_0, ..., f_n$ can all vanish at not more than a single point of $[\alpha, \beta]$. Also, the number of points x in $[\alpha, \beta]$ for which there exists $x'\neq x$ with $u(x)=u(x')$ and $w(x), w(x')\neq 0$ is finite. It can be easily seen that here u has at most two points x of discontinuity: at one of them $f_0, ..., f_n$ vanish while at the other the one-sided limits of u are ∞ and $-\infty$ (unless, of course, $x=\alpha$ or $x=\beta$).

ACKNOWLEDGMENTS

I would like to express my gratitude to Professor G. A. Freiman and to Professor B. L. Granovsky for the fruitful discussions I had with them.

REFERENCES

1. G. A. FREIMAN, "Foundations of a Structural Theory of Set Addition," Amer. Math. Soc., Providence, RI, 1973.
2. B. L. GRANOVSKY, Moment spaces of minimal dimension, *J. Approx. Theory* **49** (1987), 390–397.
3. B. L. GRANOVSKY AND E. PASSOW, Chebyshev systems of minimal degree, *SIAM J. Math. Anal.* **15** (1984), 166–169.
4. S. KARLIN AND W. J. STUDDEN, "Tchebycheff Systems: With Applications in Analysis and Statistics," Interscience, New York, 1966.
5. J. KIEFER AND J. WOLFOWITZ, Optimum designs in regression problems, *Ann. Math. Stat.* **30** (1959), 271–294.
6. E. PASSOW, Alternating parity of Tchebysheff systems, *J. Approx. Theory* **9** (1973), 295–298.